

# The Notion of a Random Gauge and its Interpretation

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## ABSTRACT

We introduce the concept of a random gauge. We propose two distinct types of random gauge that can be defined based on the concept of phase noise in scattering theory. In the context of quantum physics, we discuss a variety of possible realizations of this concept that can be connected to various Aharonov-other effects and make some connections with relativity as well.

## 1. INTRODUCTION

The guiding principle of Einstein's principle of general relativity is that the laws of physics should be expressed in a form that is independent of the coordinate frame in which one might choose to express them. Weyl generalized this concept in the realm of quantum physics by the Dirac operator for a field by introducing a Riemann connection for space-time, so that the partial derivative is replaced with a covariant derivative which yields a new Dirac operator<sup>9</sup> :

$$\bar{\partial}\Psi + \frac{1}{4}\Gamma_{ik}^j\gamma^i\gamma^k\Psi. \quad (1)$$

Note  $\Psi$  is the field, while  $\Gamma\Psi$  is the *interaction* term that tells us how the gravity field interacts with the electron. When one can neglect the gravitational interaction, one can consider a change of field coordinates  $\Psi$  rather than space-time coordinates, e.g. *gauge transformations*. (For a history, this summary is based upon the overview by Gross<sup>13</sup>.)

Quantum mechanics assigns a physical meaning to the absolute value  $|\Psi^\alpha|$ , however it does not assign a meaning to the phase of  $\Psi^\alpha$ , so  $\Psi^\alpha = |\Psi^\alpha|e^{i\theta_\alpha}$ , and  $\theta_\alpha$ , the *phase*, has always been taken to have no physical meaning. Both the Dirac equation and Lagrangian are invariant under transformations of the form:

$$\Psi \longmapsto e^{i\theta_\alpha}\Psi. \quad (2)$$

This is a *global* transformation since  $\alpha$  is constant. A global transformation is an invariance of the Dirac equation. Besides a global transformation, physicists have also considered the effect of local transformations of  $\alpha$ , local in this case means that  $\alpha = \alpha(x)$  so it depends on the particular space-time coordinates. When coordinates are included in  $\alpha$ , the Dirac equation and Lagrangian no longer are invariant under transformations because a  $d\alpha$  appears in the transformation equations. Thus, one could conclude that there is a background field that interacts with the electron. One can go through the usual argument that associates a change of frame with a complex line bundle which leads naturally to the association of the background field with electromagnetism and yet another Dirac operator:

$$\bar{\partial}_A \triangleq \gamma^i(\partial_i + \omega_j) = \bar{\partial} - ie\gamma^i A_j. \quad (3)$$

Quantum mechanics mandates that  $A$  be considered as a separate field, so one gets a new Lagrangian associated with  $A$ . This brief review of the power of invariance associated with the gauge concept now allows us to consider another type of invariance that previously seems to have been over looked.

A gauge transformation can often be associated with the evolutionary path of a quantum mechanical object. This is not observable directly, a difference can be such as when a quantum mechanical object can take distinct paths, and then be recombined to demonstrate a phase difference such as in the Aharonov-Bohm effect (A-B effect). This has been interpreted as a geometric phase in a number of papers by Aharonov and various

collaborators. Aharonov's career can be viewed as largely devoted to elucidating what phase means in quantum mechanics. The wavefunction is invariant under  $U(1)$  gauge transformations, so the wavefunction can be written with a constant relative phase function,  $\Lambda$ , so we have:

$$\Psi(x, t) \rightarrow e^{i\Lambda} \Psi(x, t). \quad (4)$$

This can be interpreted as associating a memory with a path. In general, the phase can depend upon the topological features of the phase such as associated with the A-B effect<sup>2</sup>. There is no reason, however, to limit  $\Lambda$ , it can be an arbitrary function  $\Lambda(x)$  or it could also be considered a random variable with an underlying probability density function (PDF).

## 2. RANDOM GAUGE TRANSFORMATION

Under some circumstances this "randomization" of  $\hat{\Lambda}$  could be used to explicitly bring in randomness associated with a path with an underlying PDF. Thus, it is possible to define a "random gauge" in quantum mechanics. Random path-dependent effects become a stochastic memory associated with the quantum mechanical system. We explore this in some detail and suggest an experiment to test the effects of randomness on path dependent phase difference effects.

Since phase has no physical meaning, there is no reason to associate it with a fixed parameter  $\theta_\alpha$ , instead one could assume that phase to be a random variable  $\hat{\theta}_\alpha$  with an underlying PDF  $P_{\theta_\alpha}(\theta)$ . Then one could claim that  $\Psi$ , when interpreted as either a field coordinate or  $|\Psi^\alpha|$ , has the usual Born probability interpretation in conventional quantum mechanics and must be invariant under a random gauge transformation:

$$\hat{\Psi} \mapsto e^{i\hat{\theta}_\alpha} \Psi. \quad (5)$$

This is a new type of invariance to consider, for both the Schrodinger equation or the equations of quantum field theory. This type of transform is define to be a **type 1 random gauge transformation**. If one associates randomness with gauge invariance rather than with  $\Psi$  directly, it opens some alternative ways to think about the interpretation of quantum mechanics, which will be examined in a subsequent paper.

One is immediately led to the question of meaning for this **type 1** random gauge transformation. One concept of random phase can be connected to the idea of what a "path" could mean in quantum mechanics. Empirically a path is associated with the propagation of disturbance which has an intensity pattern that can localized to a measurement device located at a point  $x$  and a time  $t$ . It is given that propagation of a disturbance within that medium depends on the positions of the point and the direction of propagation of the disturbance. Based on the wave nature of both mechanical and optical waves, a principle was proposed by Huygens' (and latter others) that explains how wave fronts propagate in time<sup>5</sup>. *Huygens' Principle* underpins the analysis of classical propagation problems in electromagnetic and acoustics<sup>7</sup>, and much of modern physics<sup>10</sup>. Empirically, the propagation of disturbances at a given point in a medium depends on the position of the point and the direction of propagation of the disturbance. This means that<sup>11</sup>:

1. *Each point can be in one of two states: excited or at rest. No concept of intensity of the distribution needs to be introduced.*
2. *If a disturbance arrives at the point  $P$  at the time  $t_2$ , then when starting at time  $t_1$  ( $t_2 \geq t_1$ ), the point  $P$  itself serves as a source of disturbances propagating in the medium.*

The mathematical form of Huygens' Principle can be expressed in a number of different ways. We choose the following:

**DEFINITION 2.1.** (*Huygens' Principle*): *Given a wave front  $\Psi(x, t)$ , it is the propagator of small waves which collectively make up the next wave front. The mathematical form of this statement is:*

$$\Psi(x, t_2) = \int G(x, y) \Psi(y, t_1) dy. \quad (t_2 > t_1) \quad (6)$$

$G(x, y) \doteq \text{kernel of the propagator.}$

The intensity pattern of the path at a latter time is the superposition of all possible paths from  $x$  to  $y$ . Now since there is no reason to choose a particular path, we can associate it with the random gauge transformation  $\hat{\theta}_\alpha$  so we have  $\hat{\theta}_\alpha = \hat{\theta}_\alpha(x)$ . Thus, because there is a random gauge associated with  $\Psi$ , there is an underling PDF associated with it as well. Our generalization is based on some work in scattering theory and other applied physics applications. Methods of determining the probability density function (PDF) are well known for the transformation:

$$\hat{x} = a \sin \hat{\theta} \quad (7)$$

where  $a$  is a deterministic parameter and  $\hat{\theta}$  is random variable with a uniform PDF<sup>17</sup>  $f_\theta(\hat{\theta})$ . The PDF's for this distribution and others are found in the Appendix. Given the characterization of the probability density function of the angle it is possible to determine the moments of the random variable:

$$\hat{y}_c = \sum_{i=1}^N \hat{r}_i \cos \hat{\theta}_i \quad (8)$$

or

$$\hat{y}_s = \sum_{i=1}^N \hat{r}_i \sin \hat{\theta}_i \quad (9)$$

for an arbitrary  $\hat{r}_i$  and  $\hat{\theta}_i$  provided their characteristic functions exist<sup>7</sup>. (Note these sums could also be viewed as discrete random Fourier series<sup>1416</sup>.) Now, the two sums can be written as:

$$\hat{z} = \hat{y}_c + i\hat{y}_s = \sum_{j=1}^N \hat{A}_j e^{i\hat{\theta}_j} \quad (10)$$

which is the same mathematical form as the random gauge transformation  $\hat{\Psi} \mapsto \sum_\alpha \hat{A}_\alpha e^{i\hat{\theta}_\alpha} \Psi$ . Thus we have a means for associating different types of randomness with the phase variable; some examples of such probability distributions are the uniform distributions, Cauchy distributions, Normal distributions, etc. In general, the mean  $\langle \cdot \rangle$  of  $\hat{z}$  can be written as:

$$\begin{aligned} \langle \hat{z} \rangle &= \langle \hat{y}_c \rangle + i \langle \hat{y}_s \rangle = \sum_{j=1}^N \langle \hat{A}_j \rangle \langle e^{i\hat{\theta}_j} \rangle \\ &= \sum_{j=1}^N \langle \hat{A}_j \rangle \langle \cos \hat{\theta}_j \rangle + i \sum_{j=1}^N \langle \hat{A}_j \rangle \langle \sin \hat{\theta}_j \rangle \end{aligned} \quad (11)$$

provided the random variables in the sums are uncorrelated. (The appendix considers specific PDF's for these angular variables.) Also, we have:

$$\begin{aligned} \text{Var}(\hat{z}) &= \text{Var}(\hat{y}_c) + i\text{Var}(\hat{y}_s) \\ &= \sum_{j=1}^N \text{Var}(\hat{A}_j) \text{Var}(\cos \hat{\theta}_j) + i \sum_{j=1}^N \text{Var}(\hat{A}_j) \text{Var}(\sin \hat{\theta}_j). \end{aligned} \quad (12)$$

Thus, the mathematics of scattering theory can be drawn upon to explore how the variances in the the paths between two paths can vary for different types of random gauges. For example, most of the examples of Berry phase can be readily generalized by exploring the variation in paths as instances of random gauges. In the next section, we consider how to demonstrate this using a Berry phase device. We then discuss how this might be related to variation in the space-time metric as an instance of aspects of gravity as a gauge transformation.

It is possible to convert the  $\sum_{i=1}^N r_i \cos \hat{\theta}_i$  to a polynomial function  $g(\theta, \varphi)$  using trigonometric identities:

$$\hat{G}(\hat{\theta}, \hat{\varphi}) = \sum_{j=1}^N r_j \cos \hat{\theta}_j. \quad (13)$$

This formula can be interpreted as an antenna pattern, since any polynomial function of variables,  $\theta$  and  $\varphi$  with  $G(\theta, \varphi)$  representing the gain of the antenna. In classical electromagnetics, an antenna is the source to launch an electromagnetic wave into space, and the antenna pattern characterizes how the electromagnetic wave are distributed in the angular variables. This suggests a second type of random gauge transformation.

A **Type 2 random gauge transformation** associates the randomness with the field coordinates so we have a family of PDF's associated with a given gauge transformation:

$$\hat{\Psi} \mapsto \exp(i\theta(\alpha(\hat{x}))) \Psi. \quad (14)$$

It is in this form that Huygens' principle can be interpreted as a statement in terms of antenna theory. Note this is done by interpreting  $G(x, y)$  as the antenna gain, which can be written as a function of angle variables:

$$G = G(\theta, \vartheta) = f(\sin \theta, \cos \vartheta) \quad (15)$$

Now each point on a spherical surface of propagation can be treated as a point source of outward propagating radiation, which, in the language of antenna theory, is equivalent to the statement that the gain of the antenna is  $G(\theta, \varphi) = \frac{1}{2}$ . If we assume that  $G(\theta, \varphi) \neq \frac{1}{2}$ , then radiation expands non-uniformly. Thus we have non-radial expansion of radiation from a point source. Huygens' principle with the gain equal to one (inward and outward expansion) was more or less an observation rather than an experimental fact or a theoretical principle etched in stone. Thus, in line with the antenna theory interpretation one might argue that a non-spherical expansion should be considered, so this would have the functional form:

$$G(\theta, \vartheta) = \sum_{j=1}^m \sum_{i=1}^n a_i \cos \vartheta_i \sin \theta_j. \quad (16)$$

The type 2 random gauge can be viewed as a generalization of the concept of an antenna gain pattern. The field coordinates in quantum field theory can be used as the source, in the same way the gain pattern of an antenna is, of random fields that arise in space time.

### 3. PHYSICS INSTANCES OF RANDOM GAUGE TRANSFORMATIONS

The type 2 random gauge concept can be associated with the Feynman's path integral and by using the random gauge concept in Huygens' principle. Given a wave front  $\Psi(x, t)$ , which is the propagator of quantum waves, the gauge acts collectively as a source to constitute the next wave front. The mathematical form of this statement is:

$$\Psi(\vartheta, t_2) = \int G(\hat{\theta}, \hat{\vartheta}) \Psi(\theta, t_1) d\theta. \quad (t_2 > t_1) \quad (17)$$

$G(\hat{\theta}, \hat{\vartheta}) \doteq$  random antenna pattern which propagates  $\Psi(\theta, t_1)$  from the time  $t_1$  to the time  $t_2$ . The point of this is the randomness we normally associate with  $\Psi$  in quantum mechanics can instead be associated with mechanism for propagating  $\Psi$  in time if the most general form of gauge invariance is supposed for quantum mechanics. It also provides a physical interpretation of the path integral by a natural connection with Huygens' principle becoming the foundational principle for quantum mechanics. Furthermore, if one takes this definition of a propagator, then one can use it to explain the gauge invariance of  $\Psi$ , and use this fact to explain phase invariance for many of the different "paths" that can be taken in quantum mechanics as well as such things as the Aharonov-Bohm effect and Berry phase.

The observation of Aharonov-Bohm phase<sup>2</sup> suggests to one a means of both interpreting and demonstrating some aspects of the random gauge concept. In the Figure of the AB-effect, the electron wave function can be written as<sup>18</sup> :

$$\Psi(\mathbf{r}, t) = \Psi_1(\mathbf{r}, t) e^{iS_1} + \Psi_2(\mathbf{r}, t) e^{iS_2}, \quad (18)$$

where the numbers 1 and 2 refer to the slits and the phases for each path the electron can take:

$$S_{1,2} = \frac{e}{\hbar c} \int_{x_0}^{x_1} \mathbf{A}_{1,2} \cdot d\mathbf{l}_{1,2}. \quad (19)$$

The phase difference between the two components of the wave function is:

$$S_1 - S_2 = \frac{e}{\hbar c} \oint_C \mathbf{A} \cdot d\mathbf{l} = \frac{e\Phi}{\hbar c} \quad (20)$$

where  $C$  is a closed contour about the solenoid and the magnetic flux  $\Phi$  is the cumulative amount of flux in the interior of the solenoid. The solenoid has a current passing through it. Instead of considering a normal current that induces AB-phase difference, which assumes a current which is proportional to  $NI_0 \cos \omega_0 t$ , where  $N$  is the number of turns in the solenoid and  $\omega_0$  is the frequency of the current.

If we consider the frequency no longer fixed, but disturbed by noise, it is evident how noise can manifest itself in the AB-effect. If we specifically modulate the current with noise,  $\hat{I} = I_0 \cos(\omega_0 t + \hat{n})$ , the effect of the noise is made manifest by an additional phase difference in the phase difference between the two components of the wave function. We have induced an effective random variation in the gauge path that is equal to:

$$\Delta \hat{S} = \hat{S}_1 - \hat{S}_2 = \frac{e\Phi \exp(i\hat{\theta})}{\hbar c}. \quad (21)$$

The expected value of  $\Delta \hat{S}$ ,  $E[\Delta \hat{S}]$ , is proportional to  $E[\exp(i\hat{\theta})]$ . For many, but not all, probability distributions in the appendix,  $E[\exp(i\hat{\theta})] = 0$ . The probability distributions for which  $E[\exp(i\hat{\theta})] \neq 0$ , are those for which the characteristic function of the probability distribution of  $\theta$  is not even. The variance in the phase difference between the two components of the wave function is:

$$Var[\Delta \hat{S}] = \frac{e\Phi}{\hbar c} [Var(\cos \hat{\theta}) + i Var(\sin \hat{\theta})]. \quad (22)$$

Thus, for zero mean Gaussian noise, the variance in the phase difference between the two components of the wave function is  $Var[\Delta \hat{S}] = \frac{e\Phi}{\hbar c} [(1 - e^{-2\sigma_\theta^2})(1 + i)]$  while for the Cauchy distribution, it is  $\frac{e\Phi}{\hbar c} [(1 - e^{-2\alpha})(1 + i)]$ . Higher order corrections can be found as well using the material in the appendix. The same argument would also hold for both the Aharonov-Casher effect and the gravitational equivalent to the AB-effect since the phase difference between two paths for the quantum phenomena "o" can be written as:

$$S_1 - S_2 = \frac{\alpha \Phi_{\text{"o"}}}{\hbar}, \quad (23)$$

where  $\alpha$  is a phenomenological constant appropriate to the particular situation and  $\Phi_{\text{"o"}}$  is the flux associated with the phenomena "o". An effective random variation in the gauge path for the phenomena "o" is equal to:

$$\Delta \hat{S} = \frac{\alpha \Phi_{\text{"o"}} e^{i\hat{\theta}_{\text{"o"}}}}{\hbar}. \quad (24)$$

A third idea is couched in general relativity and related the space-time metric. In general, the spatial metric of space-time can be expressed in terms of direction cosines, for example in three dimensional space the parameterization is:

$$\hat{x} = r \cos(\hat{\theta}) \sin(\hat{\varphi}), \quad (25)$$

$$\hat{y} = r \sin(\hat{\theta}) \sin(\hat{\varphi}), \quad (26)$$

$$\hat{z} = r \cos(\hat{\varphi}). \quad (27)$$

Now,

$$s^2 = x^2 + y^2 + z^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2 = \hat{s}^2$$

so the distance is invariant with respect to a replace of the angles by random variables, so:

$$s \rightarrow \hat{s} \rightarrow s e^{i\sigma(\theta, \varphi)}. \quad (28)$$

where  $\sigma$  is an arbitrary function of  $\theta$  and  $\varphi$ , so  $\sigma = \sigma(\theta, \varphi)$ . Therefore, it is possible to associate a fluctuation of the space-time metric random variations of in the angular variables. These random variations are equivalent to a random gauge, which can be observed using the idea of a path length phase difference for any quantum phenomena. Any phenomena which has an invariant which it is possible to replace  $s \rightarrow \hat{s}$  as in Eq (27) potentially has a hidden random gauge associated with it, so the formula for a gauge path for the phenomena "o" can be used to find it:

$$\Delta \hat{S} = \frac{\alpha \Phi_{\text{"o"}} e^{i\hat{\theta}_{\text{"o"}}}}{\hbar}. \quad (29)$$

A Mach-Zenhender interferometer, combined with weak amplification device such as has been discussed by Aharonov<sup>3</sup>, might be used to detect such random variations in the metric. In a latter publication, the random gauge concept will be used to reexamine what Aharonov has termed modular momentum.

#### 4. DISCUSSION AND CONCLUSIONS

Two concepts of random gauge invariance have been introduced in this paper with separate examples of what random gauge could mean in a physical setting. Random phase probability density functions are explained in the appendix with examples of a variety of probability density functions. One concept of random phase is associated with the Feynman's path integrals and used to provide another interpretation of the Feynman path integral. Another interpretation of random gauge is introduced using the Aharonov-Bohm effect and a method is proposed for detecting the randomness is proposed. Additionally, a gauge proposed which provides an explanation for fluctuations in the space-time metric and a method is proposed for detecting it.

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#### APPENDIX A. APPENDIX: SUPERPOSITION OF RANDOM SINUSOIDAL FUNCTIONS

The problem of random flights, dates back to a paper by Lord Rayleigh. (Note these sums could be viewed as random Fourier series<sup>14, 16</sup>.) Our generalization is based on some work in scattering theory and other applied physics applications. Methods of determining the probability density function (PDF) are well known for the transformation:

$$\hat{x} = a \sin \hat{\theta} \quad (30)$$

where  $a$  is a deterministic parameter and  $\hat{\theta}$  is a random variable with a uniform PDF<sup>17</sup>  $f_{\theta}(\hat{\theta})$ . Given the characterization of the probability density function of the angle it is possible to determine the moments of the random variable

The transformation  $\hat{z} = A \sin(\hat{\varphi})$  a PDF  $f_\varphi(\varphi)$  is onto but not one-to-one over the interval beyond  $[-A, A]$ . Thus it has an infinite number of zeros. It is more convenient to determine the characteristic function (CF) directly, so the Fourier transformation of the PDF is:

$$M_z(\omega) = \left\langle e^{i\omega A \sin(\varphi)}, f_\varphi(\varphi) \right\rangle. \quad (31)$$

The exponential can be written as:

$$\sum_{n=-\infty}^{\infty} J_n(\omega A) e^{in\varphi} = e^{i\omega A \sin(\varphi)}, \quad (32)$$

so the CF is given by:

$$M_z(\omega) = \sum_{n=-\infty}^{\infty} J_n(\omega A) F(n); \quad (33)$$

where  $F(n)$  is the Fourier transform of the PDF for the angle variable  $f_\varphi(\varphi)$  which is evaluated for  $n$ . Noting the Bessel functions can be rewritten as  $(J_{-n}(x) = (-)^n J_n(x))$ .

PROPOSITION A.1. *The CF of the transformation  $\hat{z} = A \sin(\hat{\varphi})$  is:*

$$M_z^{\sin}(\omega) = J_0(\omega A) F_s(0) + \sum_{n=1}^{\infty} J_n(\omega A) S(n) \quad (34)$$

where  $S(n) = [F(n) + (-)^n F(-n)]$ .

PROPOSITION A.2. *The CF for the transformation  $\hat{x} = A \cos(\hat{\varphi}) = A \sin(\hat{\varphi} - \frac{\pi}{2})$ , which amounts to replacing  $\varphi$  by  $\varphi - \frac{\pi}{2}$  in the exponential, so the CF is:*

$$M_z^{\cos}(\omega) = J_0(\omega A) F_c(0) + \sum_{n=1}^{\infty} J_n(\omega A) C(n). \quad (35)$$

where  $C(n) = [(-)^n F(n) + F(-n)]$ .

Expressions for the moments of the characteristic function can be found from this formula:

$$\langle x^m \rangle = \frac{1}{i^m} \left. \frac{\partial^m M_P(\omega)}{\partial \omega^m} \right|_{\omega=0} = \frac{R^m}{i^m} 2^m \{S(p)/C(p)\} J_p^{(m)}(\omega R) \Big|_{\omega=0}. \quad (36)$$

This expression allows us to determine  $\langle x^n \rangle$ , which requires us to know the  $n$ -th derivative of the Bessel function. Since  $J_0(0) = 1$  and  $J_n(0) = 0$  for  $n \neq 0$ , the only terms that remain after we take the derivative with respect to  $\omega$  and set it equal to zero are those Bessel functions that have zero coefficients, e.g. those of the form  $J_{p-m}(x)$  which are one when  $p = m$ . This allows us to determine the moments to arbitrary order using the recursion relation  $J_{p-1}(x) - J_{p+1}(x) = 2J'_p(x)$  (' denotes derivative). We can continue with this process to evaluate arbitrary derivatives of the Bessel functions to arbitrary order as:

$$\begin{aligned} 2^N J_p^{(N)}(x) = & J_{p-N}(x) + (-)^1 \binom{N}{1} J_{p+2-N}(x) + \binom{N}{2} J_{p+4-N}(x) + (-)^3 \binom{N}{3} J_{p+6-N}(x) + \dots + \\ & (-)^{N-1} \binom{N}{N-1} J_{p+N-2}(x) + (-)^N J_{p+N}(x), \end{aligned} \quad (37)$$

where  $J_p^{(N)}(x)$  denotes the  $N^{th}$  derivative of the Bessel function and  $\binom{N}{m}$  is the binomial symbol where it is understood that  $\binom{N}{m} = 0$  if  $m > N$ . Noting that  $J_0(0) = 1$  and  $J_n(0) = 0$  for  $n$  not equal to zero, then only the even derivatives of  $J_0(x)$  are not equal to zero. The second moment is for both functions are:

$$\langle x^2 \rangle = - \left. \frac{\partial^2 M_\varphi(\omega)}{\partial \omega^2} \right|_{\omega=0} = \frac{R^2}{4} [2F(0) - (F(2) + F(-2))]. \quad (38)$$

COROLLARY A.3. **Gaussian or Normal Distribution:** The PDF of the zero mean normal distribution has a CF given by:

$$\frac{1}{\sqrt{2\sigma_\theta^2\pi}}e^{-\frac{\theta^2}{2\sigma_\theta^2}} \Leftrightarrow e^{-\sigma_\theta^2\omega^2/2}$$

thus

$$F(n) = e^{-\sigma_\theta^2 n^2/2} \quad (39)$$

which is even. Note the mean is zero and the second moment for either transformation is:

$$\langle x^2 \rangle = \left. \frac{\partial^2 M_\varphi(\omega)}{\partial \omega^2} \right|_{\omega=0} = \frac{R^2}{2} [1 - e^{-2\sigma_\theta^2}]. \quad (40)$$

The third moment is zero, and the fourth moment for either transformation is:

$$\langle x^4 \rangle = \left. \frac{\partial^4 M_\varphi(\omega)}{\partial \omega^4} \right|_{\omega=0} = \frac{R^4}{8} [e^{-8\sigma_\theta^2} - 4e^{-2\sigma_\theta^2} + 3]$$

Thus, the CF's are:

$$M_\varphi^{\sin}(\omega) = M_\varphi^{\cos}(\omega) = J_0(\omega A) + 2 \sum_{m=1}^{\infty} J_{2m}(\omega A) e^{-2\sigma_\theta^2 m^2}. \quad (41)$$

COROLLARY A.4. **Non-zero mean Gaussian:** The PDF of the nonzero mean normal distribution has a CF given by:

$$\sqrt{\frac{2}{\pi^2 \sigma_\theta^2}} e^{-\omega^2 \sigma_\theta^2/2} e^{-i\omega \theta_0} \quad (42)$$

thus:

$$F(n) = e^{-n^2 \sigma_\theta^2/2} e^{-in\theta_0}. \quad (43)$$

The mean is:

$$\langle x \rangle = \left. \frac{\partial M_\varphi(\omega)}{\partial \omega} \right|_{\omega=0} = 2e^{-\sigma_\theta^2/2} [\cos \theta_0] R, \quad (44)$$

and the second moment is:

$$\langle x^2 \rangle = \left. \frac{\partial^2 M_\varphi(\omega)}{\partial \omega^2} \right|_{\omega=0} = 2 [1 - e^{-2\sigma_\theta^2} \cos(2\theta_0)] R^2, \quad (45)$$

while the fourth moments is:

$$\langle x^4 \rangle = \frac{R^4}{8} [e^{-8\sigma_\theta^2} \cos(4\theta_0) - 4e^{-2\sigma_\theta^2} \cos(2\theta_0) + 6]. \quad (46)$$

COROLLARY A.5. **Laplace Distribution:** The PDF's CF is:

$$\frac{\alpha}{2} e^{-\alpha|x|} \Leftrightarrow \frac{\alpha^2}{\alpha^2 + \omega^2}, \quad (47)$$

thus:

$$F(n) = \frac{\alpha^2}{\alpha^2 + n^2}, \quad (48)$$

which is even. Thus the mean is zero and the second moment is:

$$\langle x^2 \rangle = - \left. \frac{\partial^2 M_\varphi(\omega)}{\partial \omega^2} \right|_{\omega=0} = 2R^2 \left[ 1 - \frac{\alpha^2}{\alpha^2 + 4} \right]. \quad (49)$$



while the fourth moment is:

$$\langle x^4 \rangle = \left. \frac{\partial^4 M_\varphi(\omega)}{\partial \omega^4} \right|_{\omega=0} = \frac{R^4}{8} \left[ \frac{\alpha^2}{\alpha^2 + 16} - \frac{4\alpha^2}{\alpha^2 + 4} + 6 \right]. \quad (50)$$

COROLLARY A.6. **Cauchy Distribution:** The PDF's CF is:

$$\frac{\alpha/\pi}{\alpha^2 + x^2} \Leftrightarrow e^{-\alpha|\omega|}, \quad (51)$$

thus

$$F(n) = e^{-\alpha|n|}, \quad (52)$$

which is even. Thus we have the somewhat amusing result that the transformed Cauchy has finite moments, while the original distribution doesn't. Note the mean is zero and the second moment is:

$$\langle x^2 \rangle = - \left. \frac{\partial^2 M_\varphi(\omega)}{\partial \omega^2} \right|_{\omega=0} = 2A^2 [1 - e^{-2\alpha}]. \quad (53)$$

while the fourth moment is:

$$\langle x^4 \rangle = \left. \frac{\partial^4 M_\varphi(\omega)}{\partial \omega^4} \right|_{\omega=0} = \frac{R^4}{8} [e^{-4\alpha} - 4e^{-2\alpha} + 6]. \quad (54)$$

Thus we have the somewhat amusing result that the transformed Cauchy has finite moments, while the original distribution does not.

For products of random variables, it is still useful to know the PDF of the one dimensional sinusoidal transform. Now if we apply the identity<sup>1</sup> :

$$\int_{-\infty}^{\infty} e^{-i\omega x} J_n(x) dx = \frac{2(-i)^n T_n(\omega)}{\pi \sqrt{(1 - \omega^2)}} \Theta(1 - |\omega|). \quad (55)$$

Note  $T_n(x)$  is the n-th order Chebyshev polynomial which is defined as:

$$T_n(x) = \frac{n}{2} \sum_{m=0}^{[n/2]} (-)^m \frac{(n-m-1)!}{m!(n-2m)!} (2x)^{n-2m}, \quad (56)$$

[.] means the largest integer contained therein. Note all the subsequent properties of Chebyshev polynomials are drawn from Arfken<sup>4</sup>.

PROPOSITION A.7. The PDF of the transformation  $\hat{y} = A \sin(\hat{\theta})$  is:

$$f_y^{\sin \theta}(y) = \frac{2}{\pi} \frac{[F(0) + \sum_{n=1}^{\infty} U_{\sin(\theta)}(n) T_n(\frac{y}{A})]}{\sqrt{(1 - (\frac{y}{A})^2)}}, \quad (57)$$

where:

$$U_{\sin(\theta)}(n) = [F(n) + (-)^n F(-n)] (-i)^n. \quad (58)$$

This expression is only valid for  $y \in [-A, A]$  and zero elsewhere. Equivalently, we could multiply by  $\Theta(1 - |\frac{y}{A}|)$ .

PROPOSITION A.8. The PDF of the coordinate transformation  $\hat{y} = A \cos(\hat{\theta})$  is:

$$f_y^{\cos \theta}(y) = \frac{2}{\pi} \frac{[F(0) + \sum_{n=1}^{\infty} U_{\cos(\theta)}(n) T_n(\frac{y}{A})]}{\sqrt{(1 - (\frac{y}{A})^2)}} \quad (59)$$

where:

$$U_{\cos(\theta)}(n) = [(-)^n F(n) + F(-n)]. \quad (60)$$

This expression is only valid for  $y \in [-A, A]$  and zero elsewhere.

**COROLLARY A.9. Triangular Distribution:** Since  $F(n) = \frac{4 \sin^2(na/2)}{a^2 n^2}$ , which is an even function, the PDF's for the sin transform is given by:

$$f_y^{\sin}(y) = \frac{2}{\pi} \frac{\left[1 + 2 \sum_{m=1}^{\infty} (-)^m \frac{\sin^2(ma)}{a^2 m^2} T_{2m}\left(\frac{y}{A}\right)\right]}{\sqrt{\left(1 - \left(\frac{y}{A}\right)^2\right)}} \Theta\left(1 - \left|\frac{y}{A}\right|\right), \quad (61)$$

while that for the cosine is:

$$f_y^{\cos}(y) = \frac{2}{\pi} \frac{\left[1 + 2 \sum_{m=1}^{\infty} \frac{\sin^2(ma)}{a^2 m^2} T_{2m}\left(\frac{y}{A}\right)\right]}{\sqrt{\left(1 - \left(\frac{y}{A}\right)^2\right)}} \Theta\left(1 - \left|\frac{y}{A}\right|\right). \quad (62)$$

**COROLLARY A.10. Gaussian or Normal Distribution:** Since  $F(n) = e^{-\frac{n^2}{4\alpha}}$ , which is an even function, the PDF for the sin transformation is:

$$f_y^{\sin}(y) = \frac{2}{\pi} \frac{\left[1 + 2 \sum_{m=1}^{\infty} (-)^m e^{-\frac{m^2}{\alpha}} T_{2m}\left(\frac{y}{A}\right)\right]}{\sqrt{\left(1 - \left(\frac{y}{A}\right)^2\right)}} \Theta\left(1 - \left|\frac{y}{A}\right|\right). \quad (63)$$

The PDF for the cosine transformation is:

$$f_y^{\cos}(y) = \frac{2}{\pi} \frac{\left[1 + 2 \sum_{m=1}^{\infty} e^{-\frac{m^2}{\alpha}} T_{2m}\left(\frac{y}{A}\right)\right]}{\sqrt{\left(1 - \left(\frac{y}{A}\right)^2\right)}} \Theta\left(1 - \left|\frac{y}{A}\right|\right). \quad (64)$$

This is a Gaussian sum, so a closed form evaluation of the sum is unknown by current techniques.

**COROLLARY A.11. Laplace Distribution:** Since  $F(n) = \frac{\alpha^2}{\alpha^2 + n^2}$ , which is an even function, the PDF for the sin transformation is:

$$f_y^{\sin}(y) = \frac{2}{\pi \sqrt{\left(1 - \left(\frac{y}{A}\right)^2\right)}} \left[1 + 2 \sum_{m=1}^{\infty} (-)^m \frac{\alpha^2 T_{2m}\left(\frac{y}{A}\right)}{\alpha^2 + 4m^2}\right] \Theta\left(1 - \left|\frac{y}{A}\right|\right). \quad (65)$$

The PDF for the cosine transformation is:

$$f_y^{\cos}(y) = \frac{2}{\pi \sqrt{\left(1 - \left(\frac{y}{A}\right)^2\right)}} \left[1 + 4 \sum_{m=1}^{\infty} \frac{\alpha^2 T_{2m}\left(\frac{y}{A}\right)}{\alpha^2 + 4m^2}\right] \Theta\left(1 - \left|\frac{y}{A}\right|\right). \quad (66)$$

**COROLLARY A.12. Cauchy Distribution:** Since  $F(n) = e^{-\alpha|n|}$ , which is an even function, the PDF for the sin transformation is:

$$f_y^{\sin}(y) = \frac{2}{\pi} \frac{\left[1 + 2 \sum_{m=1}^{\infty} (-)^m e^{-2\alpha|m|} T_{2m}\left(\frac{y}{A}\right)\right]}{\sqrt{\left(1 - \left(\frac{y}{A}\right)^2\right)}} \Theta\left(1 - \left|\frac{y}{A}\right|\right). \quad (67)$$

The PDF for the cosine transformation is:

$$f_y^{\cos}(y) = \frac{2}{\pi} \frac{[1 + 4 \sum_{m=1}^{\infty} e^{-2\alpha|m|} T_{2m}(\frac{y}{A})]}{\sqrt{(1 - (\frac{y}{A})^2)}} \Theta(1 - |\frac{y}{A}|). \quad (68)$$

Further results can be obtained without considering specific PDF's. The orthogonality relationship for the Chebyshev polynomials<sup>4</sup> is:

$$\int_{-1}^1 \frac{T_m(y)T_n(y)}{\sqrt{1-y^2}} dy = \frac{\pi}{2} \delta_{m,n} (n > 0), \quad (69)$$

or  $\pi$  for  $m = n = 0$ . Any integral of a polynomial function  $f(x)$  with a Chebyshev polynomial can be evaluated using:

$$x^n = \frac{1}{2^{n-1}} [T_n(x) + \binom{n}{1} T_{n-2}(x) + \binom{n}{2} T_{n-4}(x) + \dots], \quad (70)$$

where the series terminates with  $\binom{n}{m} T_1(x)$  for  $n = 2m + 1$  or  $\frac{1}{2} \binom{n}{m} T_0(x)$  for  $n = 2m$ . With these two results, the means and standard deviations can be computed without specific knowledge of the PDF's, since a specific  $F(n)$  does not effect these calculations. The mean is:

$$\bar{x} = \frac{AU_{\sin(\theta)/\cos(\theta)}(1)}{2}. \quad (71)$$

The second moment is determined to be:

$$\bar{x}^2 = \frac{A^2}{2} \left[ 1 + \frac{U_{\sin(\theta)/\cos(\theta)}(2)}{2} \right] \quad (72)$$

The standard deviation is therefore:

$$\sigma_x = \frac{A}{\sqrt{2}} \sqrt{1 + \frac{U_{\sin(\theta)/\cos(\theta)}(2)}{2} - \frac{U_{\sin(\theta)/\cos(\theta)}^2(1)}{2}}. \quad (73)$$

The  $m^{th}$ -moment is given by:

$$\bar{x}^m = \frac{A^{2l}}{2^{2l}} [U_{\sin(\theta)/\cos(\theta)}(2l) \binom{2l}{1} U_{\sin(\theta)/\cos(\theta)}(2(l-1)) + \dots + \binom{2l}{l} F(0)], \quad (74)$$

for  $m$  even, and the  $m$ -th moment is:

$$\begin{aligned} \bar{x}^m &= \frac{A^{2l+1}}{2^{2l+1}} [U_{\sin(\theta)/\cos(\theta)}(2l+1) + \binom{2l+1}{1} U_{\sin(\theta)/\cos(\theta)}(2l-1) + \dots \\ &\quad + \binom{2l+1}{l} U_{\sin(\theta)/\cos(\theta)}(1)] \end{aligned} \quad (75)$$

for  $m$  odd.

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